Copositive Duality for Discrete Markets and Games

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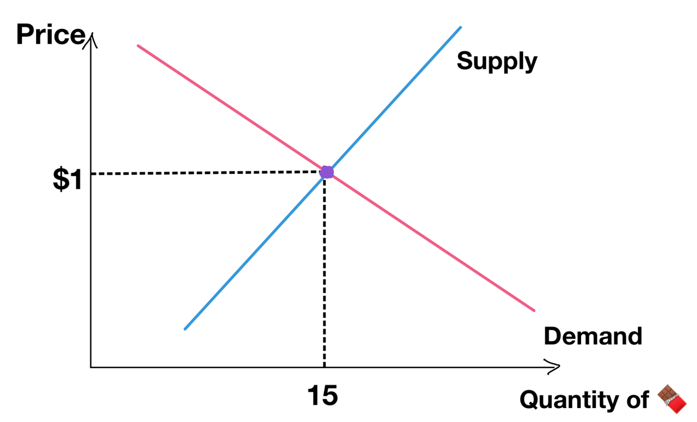
BACKGROUND

Economic Equilibrium

The Economic equilibrium is a balanced and stable state in an economic problem. Some examples include:

→ Market equilibrium: supply = demand & individually rational.



→ Nash equilibrium (NE): no deviation from equilibrium

Motivation

- Equilibrium for a convex economics problem is usually obtained by strong duality. E.g. shadow prices, Karush-Kuhn-Tucker (KKT) conditions for NE.
- For nonconvex problems with discrete decisions, strong duality generally does not exist, which is a challenge.

★ Our framework: mixed-binary quadratic programs (MBQPs) → reformulate to an equivalent convex (completely positive) program (Burer, 2009) → use strong duality of convex programs for discrete pricing and game problems.

Copositive Programming

- Copositive cone: $\mathcal{C} = \{X \in \mathcal{S} | y^\top X y \geq 0, \forall y \in \mathbb{R}^n_+\}$.
- Completely positive cone: $\mathcal{C}^* = \{XX^\top | X \in \mathbb{R}^{n \times r}, X \geq 0\}$.
- CPP (completely positive program): optimize over $X \in \mathcal{C}^*$, with objective and constraints linear on X.
- COP (copositive program): optimize over $X \in \mathcal{C}$, dual of CPP.
- In literature, CPP and COP are often solved by semidefinite program (SDP) approximations.

MBQP —Reformulation→ CPP —Relaxation→ Optimization over $\mathcal{S}^+ \cap \mathcal{N}$

CPP —Dualization→ COP

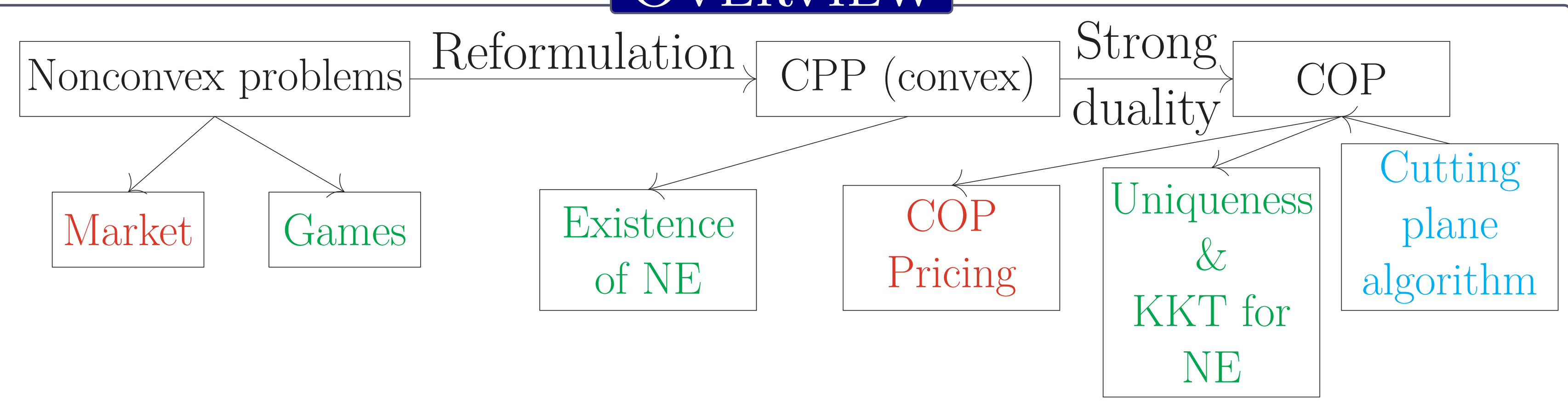
COP —Restriction→ Optimization over $\mathcal{S}^+ + \mathcal{N}$

Contributions

- A notion of duality for discrete problems
- A novel COP-based pricing scheme for nonconvex energy markets
- Theoretical results for mixed-binary quadratic games
- An exact cutting plane algorithm for mixed-integer COPs

FRAMEWORK & APPLICATIONS

Overview



Pricing in Energy Markets

Unit commitment (UC) problem: For each hour t and generator g decide

- Production levels: p_{gt}
- Commitment states: u_{gt}, z_{gt}

$$\min \sum_{g \in \mathcal{G}} \sum_{t \in \mathcal{T}} (c^p_g p_{gt} + c^u_g u_{gt})$$
$$\text{s.t.} \sum_{g \in \mathcal{G}} p_{gt} = d_t, \quad \forall t \in \mathcal{T} \quad (\lambda)$$
$$\mathbf{a}^{\phi\top}_{jgt} \mathbf{x} = b_{jgt}, \quad \forall j = [m], g \in \mathcal{G}, t \in \mathcal{T} \quad (\phi)$$
$$z_{gt} \in \{0,1\}, \quad \forall g \in \mathcal{G}, t \in \mathcal{T}$$

- Variables:
  - p_{gt}: production level
  - u_{gt}: turn on decision
  - z_{gt}: on/off status
  - $\ddot{\phi}$: slack variables
  - $\mathbf{x}^\top = (\mathbf{u}^\top, \mathbf{z}^\top, \mathbf{p}^\top, \ddot{\boldsymbol{\phi}}^\top)$
- Constraints:
  - Demand constraints
  - Operational constraints

- Traditional pricing method: restricted pricing (RP)

Solve the UC problem → Fix binary variables → Re-solve new linear program → Get dual prices (λ^*)

- Dual price λ^* → demand constraints

☹ Revenue generally does not recover operational costs

- Our pricing method: copositive dual pricing (CDP)

Reformulate UC problem to a CPP → Obtain the dual COP problem → Solve the COP problem → Get dual prices ($\lambda^*, \phi^*, \Lambda^*, \Phi^*$)

- Dual prices: λ^* → demand constraints, ϕ^* → operational constraints, Λ^* → lifted demand constraints in CPP, Φ^* → lifted operational constraints in CPP
- Thanks to strong duality of CPP:

☺ Generators: Total revenue = total costs (revenue neutrality)

☺ Individual rationality holds under certain conditions

Mixed-Binary Quadratic (MBQ) Games

- An n-person MBQ game $\mathcal{G}^{\text{MBQ}} = \langle \mathcal{I}, (\mathcal{X}_i)_{i \in \mathcal{I}}, (\mathbf{x}_i)_{i \in \mathcal{I}} \rangle$
- Each player solves an MBQP → reformulate to a CPP
- An MBQ game is transformed to a completely positive (CP) game
- With certain conditions, NE of an MBQ game ⇔ NE of a CP game

☺ Propose existence and uniqueness conditions of NE for MBQ games

☺ Obtain NE of an MBQ game via KKT conditions of the CP game

- Special case: all variables are binary + all constraints are equalities (e.g. bimatrix games) → KKT conditions can be reformulated to a single (mixed-integer) COP

ALGORITHM & NUMERICAL RESULTS

Cutting Plane for COPs

- In literature, COPs are often solved with SDP approximations
- We propose a novel cutting plane algorithm for mixed-integer COP problems.

The algorithm:

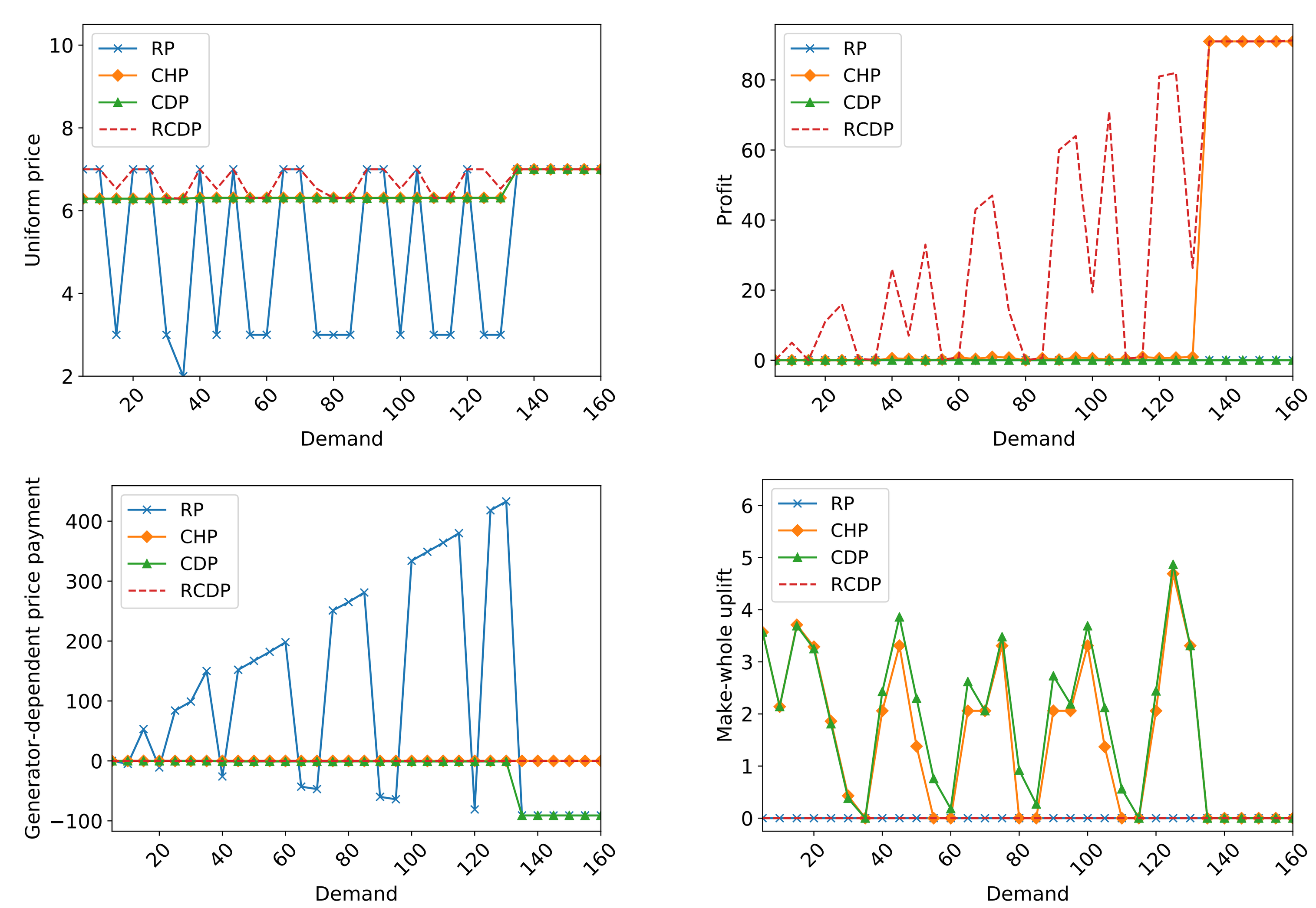
- Solve a relaxed problem without the copositive conic constraint $\Omega \in \mathcal{C}$
- Solve an MIP (Anstreicher, 2020) to separate the optimal $\hat{\Omega}$:

$$\begin{aligned} \max \quad & w \\ \text{s.t.} \quad & \hat{\Omega}\mathbf{z} \leq -w\mathbf{1} + M(1-\mathbf{u}) \\ & \mathbf{1}^\top \mathbf{u} \geq q \\ & w \geq 0 \\ & 0 \leq \mathbf{z} \leq \mathbf{u} \\ & \mathbf{u} \in \{0,1\}^{n_c} \end{aligned}$$

- If optimal $\bar{w} > 0$, add the cut $\bar{\mathbf{z}}^\top \Omega \bar{\mathbf{z}} \geq 0$ to the relaxed problem

Pricing Experiment: Scarf's Example

- Use the Scarf's example, a classical nonconvex market example, to compare RP, convex hull pricing (CHP), CDP, Revenue-adequate (RCDP)
  - CHP (Hogan and Ring, 2003): obtain prices from the Lagrangian dual problem of the UC problem
  - RCDP: add constraints to the dual COP problem to ensure *individual* revenue adequacy



Game Experiment: Bimatrix Games

- Use Bimatrix games for testing the KKT conditions
  - Converges pretty fast
  - State-of-the-art bimatrix game algorithm is faster
  - Our method is more general, can be applied to other games

Size	Time (sec)	# Iterations
2 × 2	1.54	3
3 × 3	1.48	1
4 × 4	2.45	31
5 × 5	4.75	62