Tightening Quadratic Convex Relaxations for the AC Optimal Transmission Switching Problem

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Alternating Current Optimal Transmission Switching (ACOTS)

- ACOTS: switch off lines to reduce line congestion, subject to AC power flow constraints

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ACOTS Background

- Reduce congestion without building new lines
- Similar to Braess's paradox in transportation networks
- Incorporating switching decisions to the AC optimal power flow (ACOPF) problem

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A Simplified ACOTS Formulation

$$
\begin{aligned}
\min \quad & f(p_i) && && \rightarrow \text{Convex quadratic cost function}\\
\text{s.t.} \quad & S_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^* z_{ij} && \forall (i,j) \in \mathcal{A} && \rightarrow \text{Ohm's law}\\
& S_i = \sum_{(i,j)\in\mathcal{A}\cup\mathcal{A}^R} S_{ij} && \forall i \in \mathcal{N} && \rightarrow \text{Kirchhoff's law (flow balance)}\\
& \underline{\boldsymbol{v}}_i \le |V_i| \le \overline{\boldsymbol{v}}_i && \forall i \in \mathcal{N} && \rightarrow \text{Voltage magnitude limits}\\
& |S_{ij}| \leqslant \overline{\boldsymbol{s}}_{ij} && \forall (i,j) \in \mathcal{A}\cup\mathcal{A}^R && \rightarrow \text{Apparent power upper limit}\\
& \underline{\boldsymbol{S}}_i \le S_i \le \overline{\boldsymbol{S}}_i && \forall i \in \mathcal{N} && \rightarrow \text{Power limits}\\
& z_{ij} \in \{0,1\} && \forall (i,j) \in \mathcal{A} &&
\end{aligned}
$$

- A mixed-integer nonlinear program (MINLP)
- Need to linearize $S_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^* z_{ij}$

V_i:	AC voltage
$S_{ij} = p_{ij} + \mathbf{j}q_{ij}$:	AC power flow
$S_i = p_i + \mathbf{j}q_i$:	Difference between generation and demand
z_{ij}:	Line on/off

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Challenge

$$S_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^* z_{ij}$$

- Reformulation: Lift $W_{ii} = V_i V_i^*$ and $W_{ij} = V_i V_j^*$

$$
\begin{aligned}
& S_{ij} = (W_{ii} - W_{ij})Y_{ij}^* && \forall (i,j) \in \mathcal{A} \\
& \underline{\boldsymbol{v}}_i^2 \leq W_{ii} \leq \overline{\boldsymbol{v}}_i^2 && \forall i \in \mathcal{N} \\
& W \succeq 0 \\
& \text{rank}(W) = 1
\end{aligned}
$$

- Semidefinite programming (SDP) relaxation: Remove the rank constraint $\text{rank}(W) = 1$
- Second-order cone (SOC) relaxation: 1×1 and 2×2 minors of W are nonnegative

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Quadratic Convex (QC) Relaxation

- Proposed by Hijazi, Coffrin, and Van Hentenryck (2017)
  - More efficient compared with SDP relaxation
  - Stronger compared with SOC relaxation
- Denote $W_{ii} = w_i$, $W_{ij} = w_{ij}^R + \mathbf{j}w_{ij}^I$, and $V_i = v_i\mathbf{e}^{\mathbf{j}\theta}$

$$
\begin{aligned}
w_i &= v_i^2 \\
w_{ij}^R &= z_{ij}(v_i v_j \cos(\theta_i - \theta_j)) \\
w_{ij}^I &= z_{ij}(v_i v_j \sin(\theta_i - \theta_j))
\end{aligned}
$$

- Challenge: trilinear, trigonometric, and integer terms

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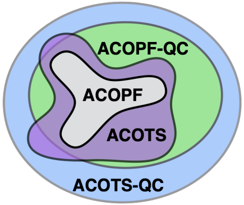
Literature Review: Convex Relaxations for ACOTS

- Became more popular recently
  - Most previous works solve ACOTS approximately: DCOTS, heuristics, etc.
- ACOTS SOC relaxation [Kocuk et al., 2017]
- ACOTS QC relaxation [Hijazi et. al. 2017]



Our Contributions

- A strengthened quadratic convex (QC) relaxation, which uses:
  - ACOTS-QC relaxation
  - Extreme-point formulation for bilinear and trilinear terms (novel for ACOTS-QC)
  - Lifted cycle constraints (novel for ACOTS-QC)
  - Optimization-based bound tightening (OBBT)

Extreme-point formulation

$+$

Cycle constraints

$+$

OBBT

ACOTS-QC Relaxation

- Quadratic function relaxation for $w_i = v_i^2$
- Let $c_{ij} = cos(\theta_{ij})$ and relax:

$$c_{ij} \leqslant 1 - \frac{1 - \cos(\boldsymbol{\theta}^{\mathbf{u}}_{ij})}{(\boldsymbol{\theta}^{\mathbf{u}}_{ij})^2}(\theta^2_{ij})$$

$$c_{ij} \geqslant \frac{\cos(\overline{\boldsymbol{\theta}}_{ij}) - \cos(\underline{\boldsymbol{\theta}}_{ij})}{\overline{\boldsymbol{\theta}}_{ij} - \underline{\boldsymbol{\theta}}_{ij}}(\theta_{ij} - \underline{\boldsymbol{\theta}}_{ij}) + \cos(\underline{\boldsymbol{\theta}}_{ij})$$

  - $z_{ij} = 0 \Rightarrow c_{ij} = 0$: use disjunctive programming
  - Similar for $s_{ij} = \sin(\theta_{ij})$

Extreme-point Formulation for Sum of Two Trilinear Terms

$$f = \boldsymbol{a}_1 w_{ij}^R + \boldsymbol{a}_2 w_{ij}^I = \boldsymbol{a}_1 v_i v_j c_{ij} + \boldsymbol{a}_2 v_i v_j s_{ij}$$

- $w_{ij}^R = v_i v_j c_{ij} \Rightarrow w_{ij}^R = \sum_{k=1}^{8} \lambda_{ijk}^c (\xi_1^k \xi_2^k \xi_3^k)$
  - $\{\xi^k\}_{k=1,\dots,8}$ are extreme points of $[\underline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_i] \times [\underline{\boldsymbol{v}}_j, \overline{\boldsymbol{v}}_j] \times [\underline{\boldsymbol{c}}_{ij}, \overline{\boldsymbol{c}}_{ij}]$
  - Linear combination of extreme points
  - Similar for $w_{ij}^I = v_i v_j s_{ij}$
- H^1 $(z_{ij} = 1)$: extreme-point formulation for f describing convex hull
  - Tighter than recursive McCormick
- H: reformulate H^1 for on-off options with big-M
- H^0 $(z_{ij} = 0)$: $c_{ij} = s_{ij} = w_{ij}^R = w_{ij}^I = 0$

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Extreme-point Formulation for Sum of Two Trilinear Terms (Continued)

Theorem

$H = \mathrm{conv}(H^0 \cup H^1)$

- An extreme-point formulation for ACOTS-QC
- Previous works: use recursive McCormick-based relaxations, not as tight

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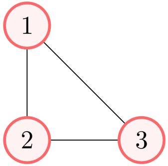
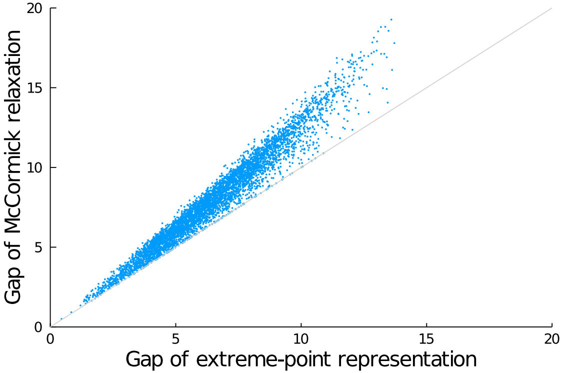
Lifted Cycle Constraints



- Strengthening constraints over cycles of power networks
  - $\theta_{ik} = \theta_{ij} + \theta_{jk} \Rightarrow cos(\theta_{ik}) = cos(\theta_{ij})cos(\theta_{jk}) - sin(\theta_{ij})sin(\theta_{jk})$
  - $c_{ik} = c_{ij}c_{jk} - s_{ij}s_{jk}, s_{ik} = c_{ij}s_{jk} + s_{ij}c_{jk}$
- Derive lifted cycle constraints on c and s
  - Previous literature: $w_j w_{ik}^R = w_{ij}^R w_{jk}^R - w_{ij}^I w_{jk}^I, \quad w_j w_{ik}^I = w_{ij}^R w_{jk}^I + w_{ij}^I w_{jk}^R$
  - ACOTS-QC + lifted cycle constraints
- Linearize bilinear terms with extreme-point formulation
- Add line-switching variables via big-M constraints
- Also implement for 4-cycles

Extreme-point Formulation vs. McCormick Relaxation

- Relaxation for summation of bilinear terms $\sum_{(j_1,j_2)\in\mathcal{P}} x^c_{j1}x^c_{j2}$, with $x^c_i \in [-1,1]$
- Gap between upper and lower bounds of the relaxation



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Cutting Plane for Lifted Cycle Constraints

- Lifted cycle constraints add extra constraints and variables
- Add only violated constraints:
  - Benders cuts from extreme point formulation (without binary variables)
  - Reformulate with disjunctive programming

☺ One separation problem per cycle; parallelizable

☺ Improved solving time for most instances

Optimization-based Bound Tightening (OBBT)

- Often used for AC power flow problems
- Tighten the bounds of v_i, θ_{ij}, z_{ij} and y_C
- Bound tightening optimization problem for variable x:

$$\begin{aligned} \text{min} / \text{max} \quad & x \\ \text{s.t.} \quad & f(p_i) \leq f^* \\ & \text{ACOTS-QC constraints with relaxed binary variables} \end{aligned}$$

Experiment: Optimality Gap Comparison

Dataset: PGLib; Language: Julia 1.6; Solver: Gurobi

- Baseline: PowerModels.jl
  - ACOTS-QC with recursive McCormick relaxation
- Compare with: ACOTS-QC + extreme-point formulation/lifted cycle constraints/OBBT
- Main findings:
  - OBBT is most helpful for SAD (small voltage angle) and API (congested) instances
  - Extreme-point formulation and lifted cycle constraints are most helpful for SAD instances

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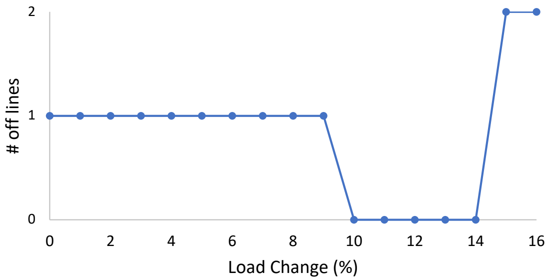
Experiment: Number of Lines Off

- Various load profiles for "TYP" case30_ieee



Experiment: Lifted Cycle Constraints for ACOPF-QC

- ACOPF-QC + lifted cycle constraints is also novel
- Most helpful for SAD instances

instances	improvement
pglib_opf_case14_ieee__sad	6.06%
pglib_opf_case24_ieee_rts__sad	0.53%
pglib_opf_case73_ieee_rts__sad	0.57%
pglib_opf_case3_lmbd__api	0.68%
pglib_opf_case24_ieee_rts__api	0.14%
pglib_opf_case73_ieee_rts__api	0.22%
pglib_opf_case179_goc__api	0.11%
nesta_case6_c__sad	0.21%
nesta_case24_ieee_rts__sad	0.53%
nesta_case29_edin__sad	4.85%
nesta_case73_ieee_rts__sad	0.57%
nesta_case118_ieee__sad	0.24%
nesta_case240_wecc__sad	0.12%
nesta_case3_lmbd__api	0.33%

Summary

- ACOTS: a MINLP problem
- Tight relaxations for ACOTS
  - ACOTS-QC relaxation
  - Extreme-point formulation
  - Lifted cycle constraints
  - OBBT